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ON THE SMALL FORCED VIBRATIONS OF SYSTEMS WITH ONE DEGREE OF FREEDOM

BY MAXIME BÔCHER

THE ordinary treatment* of the problem of the small forced vibrations of a system with one degree of freedom produced by a periodic impressed force is open to two objections. On the one hand, the solution is obtained in the form of a trigonometric series [see formulae (9) and (13) below], a form admirably adapted to certain acoustical questions, where the analysis of the vibration into its simple harmonic components is required, but poorly suited to the more fundamental purpose of enabling the investigator to follow the vibrating system in the details of its motion, since it is in general not easy to infer the nature of a function from its trigonometric development. The second objection refers to the correctness of the process by which the solution is obtained, as it is by no means clear that the trigonometric series found really represents a solution of the equation of motion.

The first of these objections was met by Radaković † who gave formulae for expressing the forced vibrations in closed form [cf. (5) and (12) below]. The second objection may also be met by imposing suitable restrictions on the function $f(x)$ which determines the impressed force.‡ It turns out, however, that for the correctness of the final result these restrictions are unnecessary. In the present paper I wish to show how the whole subject may be developed in a manner at once rigorous and extremely general. This treatment, which is thoroughly elementary, follows, in the deduction of Radaković's formulae, a method which, while not essentially different from that of Radaković, will, it is thought, be found more perspicuous. I have also considered various matters which are passed over without mention by Radaković, for instance the case in which the damping is so great that the *free* motion would not be vibratory.

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* See for instance Lord Rayleigh's *Theory of Sound*, vol. 1, § 47.

† *Wiener Sitzungsberichte. Math. Naturwiss Klasse*, vol. 114, II*a* (1905), p. 877.

‡ See Horn, *Gewöhnliche Differentialgleichungen* (*Sammlung Schubert*), §23.

The problem of the forced vibration of systems with one degree of freedom when the impressed force is periodic is equivalent to the problem of solving the differential equation

$$(1) \qquad \frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + \beta y = f(x),$$

or, more particularly, of finding the periodic solutions of this equation, where a is a constant positive or zero which measures the amount of damping, β a positive constant, and $f(x)$ a real periodic function which we will assume to be finite for all real values of x and to have at most a finite number of discontinuities in any finite interval.

We begin by considering the case where

$$(2) \qquad \beta > a^2 > 0,$$

so that the damping does not vanish and is not very great.* The general solution of the reduced equation (that is the equation obtained by replacing the second member of (1) by zero) is

$$(3) \qquad k_1 e^{-ax} \cos \sqrt{\beta - a^2}\, x + k_2 e^{-ax} \sin \sqrt{\beta - a^2}\, x.$$

For the sake of brevity we write

$$s = \sqrt{\beta - a^2}.$$

By the formula derived from the method of variation of constants (see for instance Forsyth's *Treatise*, §66) we may write the general solution of (1) in the form

$$(4) \qquad \phi(x) = k_1 e^{-ax} \cos sx + k_2 e^{-ax} \sin sx + \frac{1}{s}\int_0^x f(\xi) e^{-a(x-\xi)} \sin s(x - \xi)\, d\xi.$$

We wish now, if possible, to determine the constants of integration, k_1 and k_2, in such a way that $\phi(x)$ is periodic. It is clear that no solution of (1) can have a period which is not also a period of $f(x)$. Let us then denote by $2p$ any period of $f(x)$ and inquire whether k_1 and k_2 can be so determined that $\phi(x)$ also has the period $2p$, i. e., so that

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* This is the case in which the *free* motion is vibratory, cf. formula (3) below. If $\beta \leqq a^2$ we have for free motion the case of no oscillation illustrated by the dead-beat galvanometer.

$$\begin{aligned}
&k_1e^{-a(x+2p)}\cos s(x+2p) - k_1e^{-ax}\cos sx\\
&+ k_2e^{-a(x+2p)}\sin s(x+2p) - k_2e^{-ax}\sin sx\\
&+ \frac{1}{s}\int_0^{2p} f(\xi)e^{-a(x+2p-\xi)}\sin s(x+2p-\xi)d\xi\\
&+ \frac{1}{s}\int_{2p}^{2p+x} f(\xi)e^{-a(x+2p-\xi)}\sin s(x+2p-\xi)d\xi\\
&- \frac{1}{s}\int_0^{x} f(\xi)e^{-a(x-\xi)}\sin s(x-\xi)d\xi = 0.
\end{aligned}$$

The last two integrals here cancel on account of the periodicity of $f(\xi)$, and the equation may then be written in the form

$$\begin{aligned}
\cos sx \Big\{ &k_1(1 - e^{-2pa}\cos 2ps) - k_2 e^{-2pa}\sin 2ps\\
&\qquad - \frac{1}{s}\int_0^{2p} f(\xi)e^{-a(2p-\xi)}\sin s(2p-\xi)d\xi\Big\}\\
+ \sin sx \Big\{ &k_1 e^{-2pa}\sin 2ps + k_2(1 - e^{-2pa}\cos 2ps)\\
&\qquad - \frac{1}{s}\int_0^{2p} f(\xi)e^{-a(2p-\xi)}\cos s(2p-\xi)d\xi\Big\} = 0.
\end{aligned}$$

A necessary and sufficient condition that this equation be fulfilled for all real values of x is that the coefficients of $\cos sx$ and $\sin sx$ vanish. This gives us, for determining k_1 and k_2, two linear equations whose determinant is

$$\Delta = (1 - e^{-2pa})^2 + 4e^{-2pa}\sin^2 ps.$$

Since, under the conditions we have imposed, this quantity cannot vanish, being the sum of two real squares of which the first is not zero, we see that there is one and only one determination of k_1 and k_2, and we readily find

$$k_1 = \frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(2p-\xi)}[\sin s(2p-\xi) + e^{-2pa}\sin s\xi]d\xi,$$

$$k_2 = \frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(2p-\xi)}[\cos s(2p-\xi) - e^{-2pa}\cos s\xi]d\xi.$$

Substituting these values in (4), we find Radaković's formula

$$\phi(x) = \frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(x+2p-\xi)}\sin s(x+2p-\xi)\,d\xi \tag{5}$$
$$-\frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(x+4p-\xi)}\sin s(x-\xi)\,d\xi$$
$$+\frac{1}{s}\int_0^{x} f(\xi)e^{-a(x-\xi)}\sin s(x-\xi)\,d\xi.$$

We have thus proved

THEOREM 1. *If the impressed force has the period* $2p$ *and the damping is not zero but is sufficiently weak so that condition* (2) *is fulfilled, there exists one and only one forced vibration of period* $2p$, *and this vibration is given by* (5).

We can now see that there exists only one periodic forced vibration of any kind. For, if possible, let $\phi_1(x)$ and $\phi_2(x)$ represent two different periodic forced vibrations with periods $2p_1$ and $2p_2$ respectively. Let $2p$ denote the least common multiple of $2p_1$ and $2p_2$. Then both ϕ_1 and ϕ_2 have $2p$ as a period. This must therefore, as we have seen, be a period of $f(x)$, and therefore, by Theorem 1, there can be only one forced vibration with this period, so that our assumption has led us to a contradiction. We now easily infer

THEOREM 2. *Under the conditions of Theorem* 1, *there exists one and only one periodic forced vibration. If the impressed force is a constant, we have the degenerate case in which the function* $\phi(x)$ *which determines the vibration is a constant. Apart from this case, the periodic forced vibration has the same primitive period as the impressed force.*

Let us now consider the case in which

$$\beta < a^2 \tag{6}$$

but where a and β are still supposed to be both positive.

We will show that the analysis we have just used applies to this case also. It is true that in this case $s = \sqrt{\beta - a^2}$ is no longer real, and the trigonometric functions must be replaced by hyperbolic functions throughout. In particular we get

$$\Delta = (1 - e^{-2pa})^2 - 4e^{-2pa}\sinh^2 p\sqrt{a^2-\beta},$$

and the reasoning by which we showed that Δ cannot vanish is no longer valid. The fact, however, is still true and may be established as follows.

We may write

$$\Delta = (1 - e^{-2pa} - 2e^{-pa} \sinh p\sqrt{a^2 - \beta})(1 - e^{-2pa} + 2e^{-pa} \sinh p\sqrt{a^2 - \beta})$$
$$= 4e^{-2pa}(\sinh pa - \sinh p\sqrt{a^2 - \beta})(\sinh pa + \sinh p\sqrt{a^2 - \beta}).$$

Since none of these factors vanishes, $\Delta \neq 0$.

The remaining work which leads up to formula (5) requires no change, and formula (5) remains correct. If we wish, we may replace the imaginary quantity s in this formula by the real quantity $\sqrt{a^2 - \beta}$, at the same time replacing sin by sinh. We thus get

THEOREM 3. *Theorems* 1 *and* 2 *remain true if condition* (2) *is replaced by condition* (6).

There remains the case $\beta = a^2$. This case may be discussed along the same lines as the other two cases if we replace formula (3) by

$$k_1 xe^{-ax} + k_2 e^{-ax}.$$

Since no special difficulty and no special point of interest presents itself here, we content ourselves with merely stating the result, that Theorems 1 and 2 remain unchanged here too, provided in formula (5) the fractions

$$\frac{\sin s(x + 2p - \xi)}{s}, \qquad \frac{\sin s(x - \xi)}{s}$$

are replaced by $x + 2p - \xi$ and $x - \xi$ respectively.

We have thus carried through the discussion of all cases where there is damping.* The case $a = 0$ presents some peculiarities to which we shall come back presently. We wish, however, first to consider the question of developing the function $\phi(x)$, which determines the periodic forced vibration, in a Fourier's series

$$(7) \qquad \phi(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty}\left(a_i \cos \frac{i\pi x}{p} + b_i \sin \frac{i\pi x}{p}\right).$$

That this development converges absolutely and uniformly for all values of x to the value $\phi(x)$ follows from the fact that $\phi(x)$ is not merely itself continuous but has a continuous derivative.* The coefficients a_i and b_i, or as we will call them, following a suggestion of Hurwitz, the Fourier's constants of $\phi(x)$, might be determined by making use of the expression (5) for $\phi(x)$. They may be determined much more readily as follows by the method of undetermined coefficients.

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* In fact our discussion also covers the cases, which do not present themselves in the physical problem with which we are concerned, in which a and β may have any real values positive or negative, barring as yet the cases in which a or β vanish.

By differentiating the series (7) term by term first once and then twice, we get new trigonometric series which we will call (7′) and (7″). Although we shall not attempt to make any statement as to whether these series represent the functions $\phi'(x)$ and $\phi''(x)$, or even as to whether they converge, we do know that their coefficients are the Fourier's constants of $\phi'(x)$ and $\phi''(x)$ respectively.† If, then, we multiply the terms of (7) by β, those of (7′) by $2a$ and add the two series thus obtained to (7″), we shall get a series whose coefficients are the Fourier's constants of $\phi'' + 2a\phi' + \beta\phi$, that is of $f(x)$.‡ If we denote the Fourier's constants of $f(x)$ by γ_i, δ_i, so that

$$(8) \qquad \gamma_i = \frac{1}{p}\int_0^{2p} f(\xi)\cos\frac{i\pi\xi}{p}\,d\xi, \qquad \delta_i = \frac{1}{p}\int_0^{2p} f(\xi)\sin\frac{i\pi\xi}{p}\,d\xi,$$

we are thus led to the equations

$$\gamma_i = -\frac{i^2\pi^2}{p^2}a_i + \frac{2i\pi}{p}ab_i + \beta a_i,$$

$$\delta_i = -\frac{i^2\pi^2}{p^2}b_i - \frac{2i\pi}{p}aa_i + \beta b_i.$$

These are linear equations for the determination of a_i and b_i in terms of the known quantities γ_i, δ_i. The determinant of these equations is

$$D_i = \frac{1}{p^4}\left\{(p^2\beta - i^2\pi^2)^2 + 4i^2p^2\pi^2a^2\right\}.$$

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[*] Cf. for instance the writer's *Introduction to the Theory of Fourier's Series*, these ANNALS, vol. 7 (1906), p. 109, Theorem II; also published separately by Harvard University.

[†] This follows from the fact that ϕ and ϕ' are continuous, while ϕ'' is finite for all values of x and has only a finite number of discontinuities in any finite interval. Cf. the paper last referred to, p. 116, Theorem I.

[‡] Cf. p. 87, Theorems V, VI, of the paper last cited.

This quantity does not vanish, since $a \neq 0$, $\beta \neq 0$. Thus we get for a_i and b_i the values

$$a_i = \frac{1}{p^2 D_i}\left\{(p^2\beta - i^2\pi^2)\gamma_i - 2ip\pi a\delta_i\right\},$$

$$b_i = \frac{1}{p^2 D_i}\left\{2ip\pi a\gamma_i + (p^2\beta - i^2\pi^2)\delta_i\right\}.$$

Substituting these values in the series (7), we get the Fourier's development of $\phi(x)$. This development may, by a simple trigonometric reduction, be thrown into the form

$$(9) \qquad \phi(x) = \frac{\gamma_0}{2\beta} + \sum_{i=1}^{\infty}\left\{\frac{\gamma_i}{\sqrt{D_i}}\cos\frac{i\pi}{p}(x - x_i) + \frac{\delta_i}{\sqrt{D_i}}\sin\frac{i\pi}{p}(x - x_i)\right\},$$

where the constants x_i are to be determined by the equations

$$(10) \qquad \cos\frac{i\pi}{p}x_i = \frac{p^2\beta - i^2\pi^2}{p^2\sqrt{D_i}}, \qquad \sin\frac{i\pi}{p}x_i = \frac{2ip\pi a}{p^2\sqrt{D_i}}.$$

Thus we have established

Theorem 4. *The function which represents the periodic forced vibration when the damping does not vanish can always be developed into the absolutely and uniformly convergent Fourier's series* (9) *where* γ_i, δ_i *are the Fourier's constants* (8) *of the function* $f(x)$ *which determines the periodic impressed force. This last function may be any finite function with the period* $2p$ *which in every finite interval has at most a finite number of discontinuities, and need not be developable into a convergent Fourier's series.*[*]

The case of an undamped forced vibration ($a = 0$) still remains to be considered. Here the determinants Δ and D_i become respectively

$$(11) \qquad \Delta = 4\sin^2 p\sqrt{\beta}, \qquad D_i = \frac{(p^2\beta - i^2\pi^2)^2}{p^4}.$$

In order to insure that these determinants be different from zero, we must assume that $p\sqrt{\beta}$ is not an integral multiple of π. In other words, we assume that the period $2p$ of the impressed force is not an integral multiple of the period $2\pi/\sqrt{\beta}$ of the *free* vibration. Under this assumption our analysis, so far as it refers to Theorems 1 and 3, requires no modification. Formula (5) now easily reduces to

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[*] This result also, and its proof, applies without change when a, or β, or both are negative.

$$(12)\qquad \phi(x)=\frac{1}{2\sqrt{\beta}\sin p\sqrt{\beta}}\int_0^{2p}f(\xi)\cos\sqrt{\beta}\,(x+p-\xi)d\xi + \frac{1}{\sqrt{\beta}}\int_0^{x}f(\xi)\sin\sqrt{\beta}\,(x-\xi)d\xi,$$

a formula due to Radaković.

The Fourier's development of $\phi(x)$ also simplifies, for we see from (10) that in this case all the quantities x_i vanish, and the series for $\phi(x)$ becomes

$$(13)\qquad \phi(x)=\frac{\gamma_0}{2\beta}+\sum_{i=1}^{\infty}\left\{\frac{p^2\gamma_i}{p^2\beta-i^2\pi^2}\cos\frac{i\pi x}{p}+\frac{p^2\delta_i}{p^2\beta-i^2\pi^2}\sin\frac{i\pi x}{p}\right\}.$$

We thus obtain from Theorems 1 and 3

THEOREM 5. *If there is no damping and the impressed force has a period* $2p$ *which is not an integral multiple of the primitive period* $2\pi/\sqrt{\beta}$ *of the free vibration, there exists one and only one forced vibration of period* $2p$, *and this is given by formula* (12) *and also by the series* (13).

Theorem 2 obviously does not admit of extension to this case except in the special case in which the periods of the impressed force and of the free vibration are incommensurable.

In the special case in which $2p$ is an integral multiple of $2\pi/\sqrt{\beta}$ it will be readily seen by a reference to the linear equations for determining k_1 and k_2 that equation (1) has no solutions with the period $2p$ unless

$$\int_0^{2p}f(\xi)\cos\sqrt{\beta}\xi\,d\xi=0,\qquad \int_0^{2p}f(\xi)\sin\sqrt{\beta}\xi\,d\xi=0;$$

but that if these conditions are fulfilled, every solution of (1) has the period $2p$.

HARVARD UNIVERSITY,
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